Exchanging between symmetric circular formations of moving particles

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Introduction

Collective motion:

Applications:

Objectives:

- Particles with coupled oscillator dynamics in symmetric circular formations [1, 3, 2, 4].
- Changing from one cluster configuration to another.

Particles with coupled oscillator dynamics

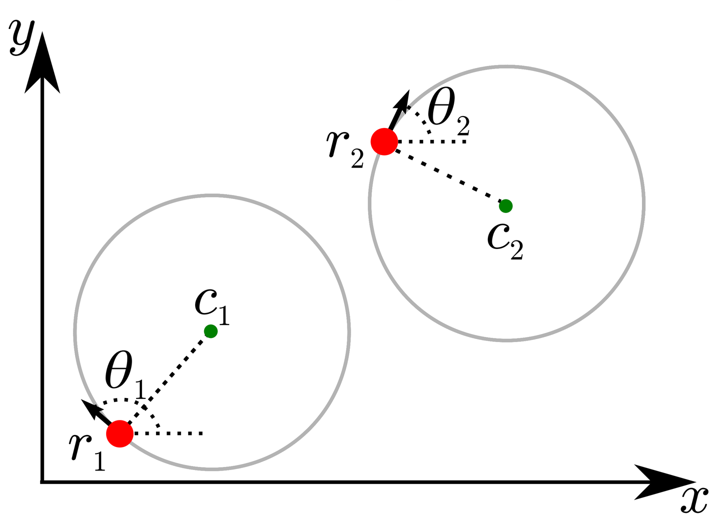
N self-propelled particles in the plane with unitary speed [2]:

$$\dot{r}_k = e^{i\theta_k} \quad \text{(1a)}$$
$$\dot{\theta}_k = u_k(\boldsymbol{r}, \boldsymbol{\theta}) \quad \text{(1b)}$$

for $k = 1, \ldots, N$.
Position: $r_k = x_k + iy_k \in \mathbb{C}$.
Direction of the speed vector: $e^{i\theta_k} = \cos\theta_k + i\sin\theta_k$.
Phase (heading angle): $\theta_k \in \mathbb{R}$.
$\boldsymbol{r} \doteq (r_1, \ldots, r_N)^T \in \mathbb{C}^N$, $\boldsymbol{\theta} \doteq (\theta_1, \ldots, \theta_N)^T \in \mathbb{T}^N$.

Rotation center of particle k, when $\dot{\theta}_k = \omega_0$:

$$c_k \doteq r_k + \omega_0^{-1} i e^{i\theta_k}. \quad (2)$$



We are interested in particles sharing the same rotation center, i.e., $c_1 = c_2 = \cdots = c_N$. It happens when $P\boldsymbol{c} = 0$, with $P = I_N - \frac{1}{N}\boldsymbol{1}\boldsymbol{1}^T$, for I_N being the $N \times N$ identity matrix, $\boldsymbol{1} = (1, \cdots, 1)$ and $\boldsymbol{c} = (c_1, \cdots, c_N)$.
The aim is to minimize the following potential:

$$S(\boldsymbol{r}, \boldsymbol{\theta}) = \frac{1}{2} \parallel P\boldsymbol{c} \parallel^2, \quad (3)$$

whose gradient is

$$\dot{S} = \langle \dot{\boldsymbol{c}}, P\boldsymbol{c} \rangle = \omega_0^{-1} \sum_{j=1}^{N} (\omega_0 - u_j) \left\langle e^{i\theta_j}, P_j\boldsymbol{c} \right\rangle, \quad (4)$$

This inner product is defined as $\langle z_1, z_2 \rangle = \mathrm{Re}\{z_1^* z_2\}$, for $z_1, z_2 \in \mathbb{C}$, and z_1^* is the conjugate of the complex number z_1.
Choosing:

$$u_k = \omega_0(1 + K_0 \left\langle e^{i\theta_k}, P_k\boldsymbol{c} \right\rangle) \quad (5)$$

leads to

$$\dot{S} = -K_0 \sum_{j=1}^{N} \left\langle e^{i\theta_j}, P_j\boldsymbol{c} \right\rangle^2 \leq 0. \quad (6)$$

Now, consider the m-th phase order parameter:

$$p_{m\theta} = \frac{1}{mN} \sum_{j=1}^{N} e^{im\theta_j} \quad (7)$$

with $0 \leq |p_{m\theta}| \leq 1/m$, which produces the phase potential:

$$U_m(\boldsymbol{\theta}) = \frac{N}{2} |p_{m\theta}|^2. \quad (8)$$

Theorem 1. *Let $1 \leq M \leq N$ be a divisor of N. Then $\boldsymbol{\theta} \in \mathbb{T}$ forms M symmetric clusters if and only if it is a global minimum of the potential*

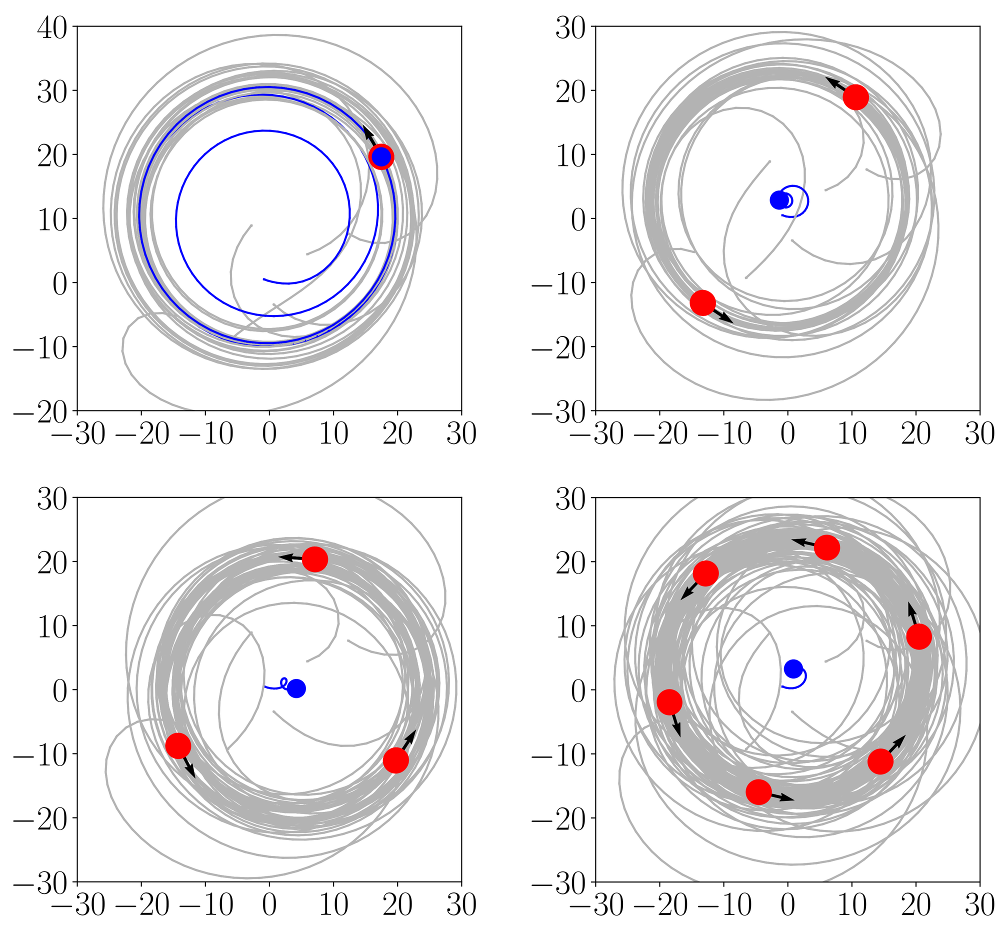
$$U^{M,N}(\boldsymbol{\theta}) = \sum_{m=1}^{M} K_m U_m \quad (9)$$

with $K_m > 0$ for $m = \{1, \cdots, M-1\}$, $K_M < 0$ (Proof in [4]).

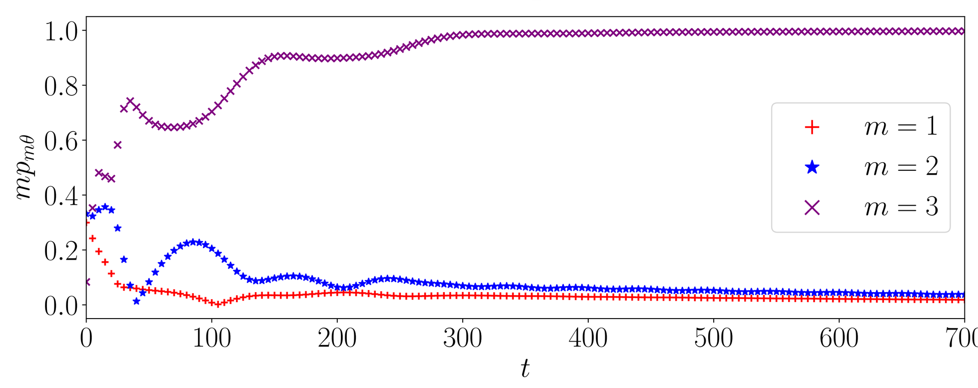
$$-\frac{\partial U^{M,N}}{\partial \theta_k} = \frac{1}{N} \sum_{m=1}^{M} \sum_{j=1}^{N} \frac{K_m}{m} \sin(m(\theta_k - \theta_j)). \quad (10)$$

$$u_k = \omega_0(1 + K_0 \left\langle e^{i\theta_k}, P_k\boldsymbol{c} \right\rangle) - \frac{\partial U^{M,N}}{\partial \theta_k}. \quad (11)$$

Model (1) with control (11) and configuration: $N = 6$, $\omega_0 = 0.05$, $K = 0.1$, $K_m = 0.18$ for $m < M$ and $K_M = -0.02$:



Time series for the order parameters ($M = 3$):

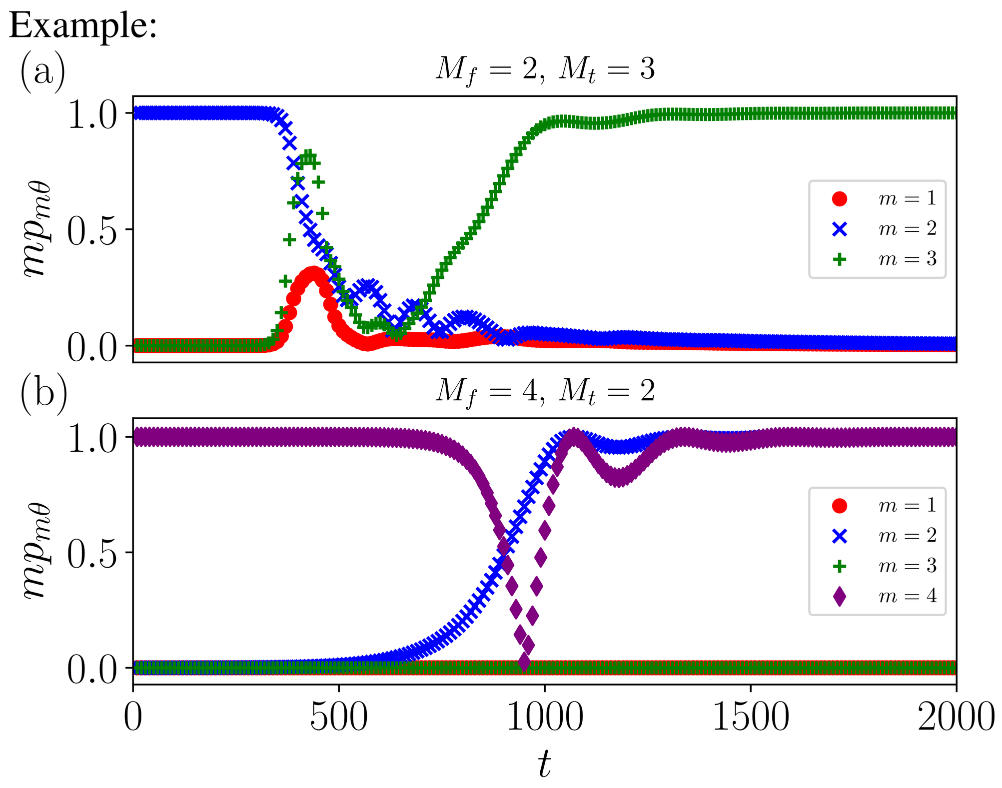


Exchanging between formations

Aim: Go from a M_f clusters configuration to M_t clusters.
The transition works perfectly in the following two situations:

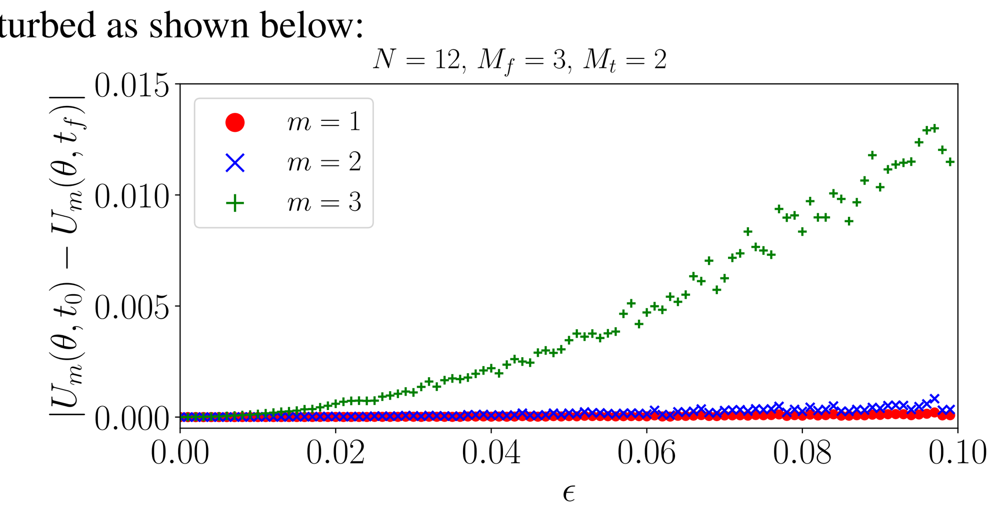
- *I)* $M_f < M_t$;
- *II)* $M_f > M_t$ when $(M_f/M_t) \in \mathbb{N}$.

Example:



Problem: In some cases, the configuration with M_f clusters may be a local minimum of the M_t so the transition never happens.

Example with $M_f = 3$ and $M_t = 2$: We start the system with 3 symmetric clusters and start the simulation for $M = 2$ clusters. The transition only works if this initial condition is strongly perturbed as shown below:



Solution: Suppress the previous configuration with the inclusion of a new term into the potential (9):

$$U_{M_f}^{M_t,N} := U^{M_t,N} + \frac{N}{2}\delta(M_f, M_t) K_1 |p_{M_f\theta}|^2, \quad (12)$$

where

$$\delta(M_f, M_t) = \begin{cases} 1, & M_f > M_t \text{ and } M_f/M_t \notin \mathbb{N} \\ 0, & \text{otherwise} \end{cases}. \quad (13)$$

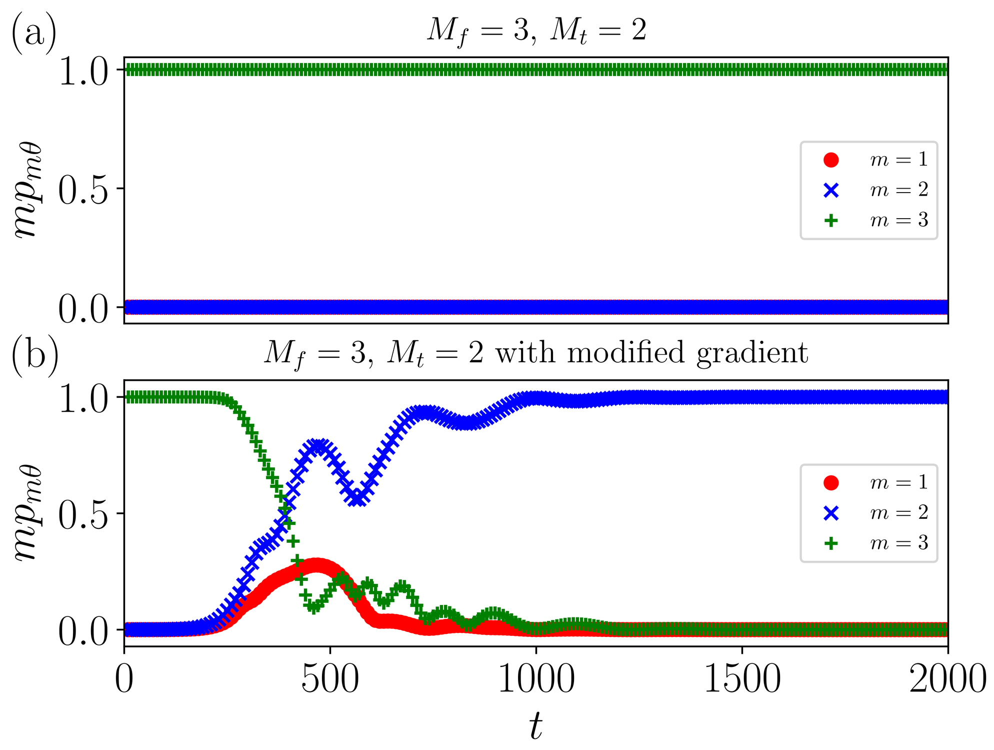
Here $K_m > 0$ for $m = \{1, 2, \cdots, M_t - 1\}$, $K_1 > 0$ and $K_m < 0$ for $m = M_t$. The new gradient reads

$$-\frac{\partial U_{M_f}^{M_t,N}}{\partial \theta_k} = \begin{aligned} &\frac{1}{N}\sum_{m=1}^{M_t}\sum_{j=1}^{N} \frac{K_m}{m}\sin(m(\theta_k - \theta_j)) \\ &+\delta\frac{1}{N}\sum_{j=1}^{N}\frac{K_1}{M_f}\sin(M_f(\theta_k - \theta_j)) \end{aligned}, \quad (14)$$

and the control

$$u_k(\boldsymbol{r}, \boldsymbol{\theta}) = \omega_0(1 + K_0 \left\langle e^{i\theta_k}, P_k\boldsymbol{c} \right\rangle) - \frac{\partial U_{M_f}^{M_t,N}}{\partial \theta_k}. \quad (15)$$

Time series of Order Parameters for model (1) with $M_t = 2$ clusters, starting from a $M_f = 3$ clusters configuration: (a) Control (11); (b) Control (15):



Acknowledgments

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