Characterization of chaos in two-dimensional dissipative discontinuous maps

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1. Abstract

The dynamical property of a dissipative two-dimensional discontinuous standard mapping are considered. The map is controlled by action and angle variables, also parameterized by two control parameters namely, k controlling the intensity of the nonlinearity and γ the dissipation. When $\gamma = 0$ the non-dissipative model is recovered while any $\gamma \neq 0$ breaks the area preservation, leading to the existence of attractors, including chaotic ones. We show that when starting from a large initial action, the dynamics converges to chaotic attractors in an exponential decay in time, while the speed of the decay depends on the dissipation intensity. We also characterize the chaotic behavior by the positive Lyapunov exponents.

2. Methods

In this work we consider the discontinuous dissipative standard mapping

$$T : \begin{cases} I_{n+1} = (1-\gamma)I_n + k\sin(\theta_n)\mathrm{sgn}[\cos(\theta_n)] \\ \theta_{n+1} = [\theta_n + I_{n+1}] \mod (2\pi) \end{cases}, \tag{1}$$

where I and θ are the action and angle variables, k and $\gamma \in [0,1]$ are parameters responsibles for the control the intensity of nonlinearity and dissipation respectively. The conservative case, obtained when $\gamma = 0$, presents a diffusive regime in the action for $k > 1$ [1, 2]. The slow and quasilinear diffusion regimes occur for $k < 1$ and $k > 1$, respectively. Figure 1 shows the evolution of an orbit for the conservative case, i.e. for $\gamma = 0$, for the initial condition $I_0 = 0.01$ and $\theta_0 = 0.01$ with (a) $k = 0.01$ and (b) $k = 10$. One can see from Figure 1(a) that, in contrast of the conservative standard map, here the mapping does not show the standard regular behavior: In this case the KAM theorem is not satisfied due to the discontinuous function in (1). Also, Fig. 1(b) shows the diffusion of a conservative chaotic orbit. Moreover, Fig. 1(c) shows the evolution of an orbit for the dissipative system using $\gamma = 10^{-2}$ for the initial conditions $I_0 = 0.01$ and $\theta_0 = 0.01$ with $k = 10$, such that diffusion of the action along the chaotic attractor is observed.

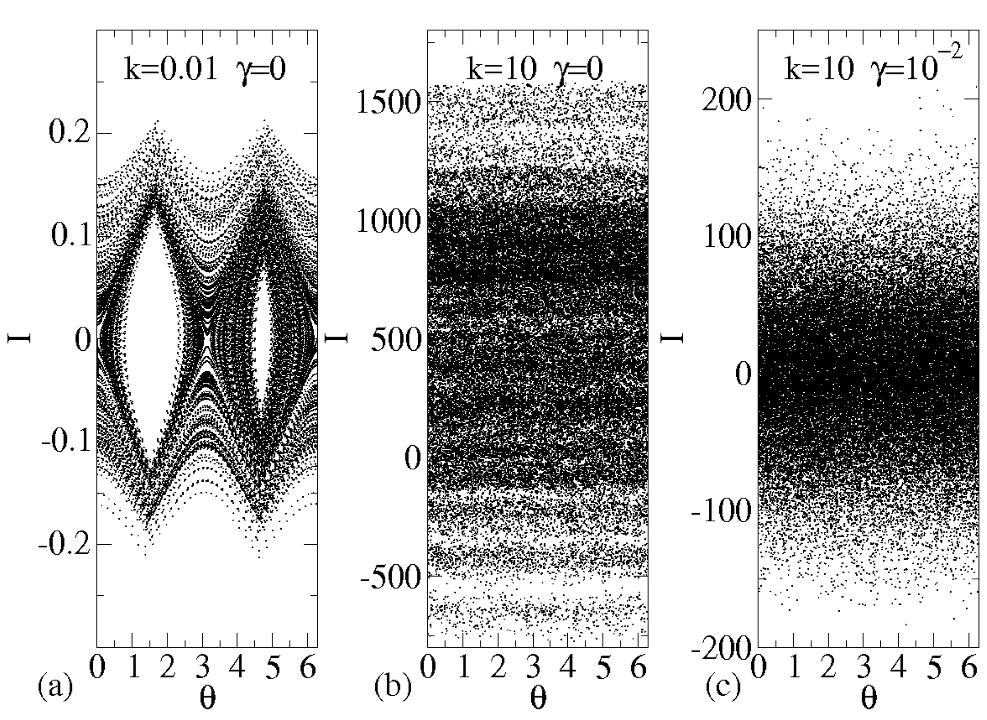

Figure 1: *Phase space for the discontinuous standard mapping (1) using the control parameters (a) $k = 0.01$ and $\gamma = 0$, (b) $k = 10$ and $\gamma = 0$, and (c) $k = 10$ and $\gamma = 10^{-2}$.*

3. Results

In this section we discuss an analytical argument for approaching orbits to the chaotic attractors. We then start with an initial condition I_0 and iterate the second equation of the mapping (1) we can obtain a generalized expression given by

$$I_n = (1-\gamma)^n I_0 + k\sum_{i=0}^{n-1}(1-\gamma)^{n-1-i}\sin(\theta_i) \tag{2}$$

Considering a small value of γ and realizing expansions in Taylor's series we can obtain the equation

$$I_n = I_0 e^{-\gamma n}, \tag{3}$$

meaning an exponential decay towards the chaotic attractors. Figure 2(a) shows the behavior of I given by mapping (1) as a function of n using $k = 10$ for different initial actions and values of γ (as labeled in the figure). The exponential fitting $I = Ae^{\mathrm{Bn}}$ (yellow curve) to the numerical data (black curve) with $\gamma = 8 \times 10^{-4}$ and $I_0 = 12 \times 10^4$ provides the fitting constants $A = 11,633 \times 10^5$ and $B = -0.000078465$. Comparing with Eq. (3) we conclude that γ corresponds well to the coefficient B. Thus, by considering the following transformations $I \to I/I_0$ and $n \to n\gamma$ we show in Fig. 2(b) the merge of the four different curves shown in 2(a) onto a single plot, therefore giving evidences that the exponential decay is scale invariant with respect to the control parameter γ as well as to the initial action I_0.

Let us now measure the Lyapunov exponents for the chaotic attractors [3, 4]

$$\lambda_j = \lim_{n\to\infty}\frac{1}{n}\ln|\Lambda_j^n|, \quad j = 1, 2, \tag{4}$$

where $\Lambda^{(n)}$ are the eigenvalues of the matrix $M = \prod_{i=1}^{n} J_i(\theta, I)$ with J_i representing the Jacobian matrix of the mapping evaluated along the orbit.

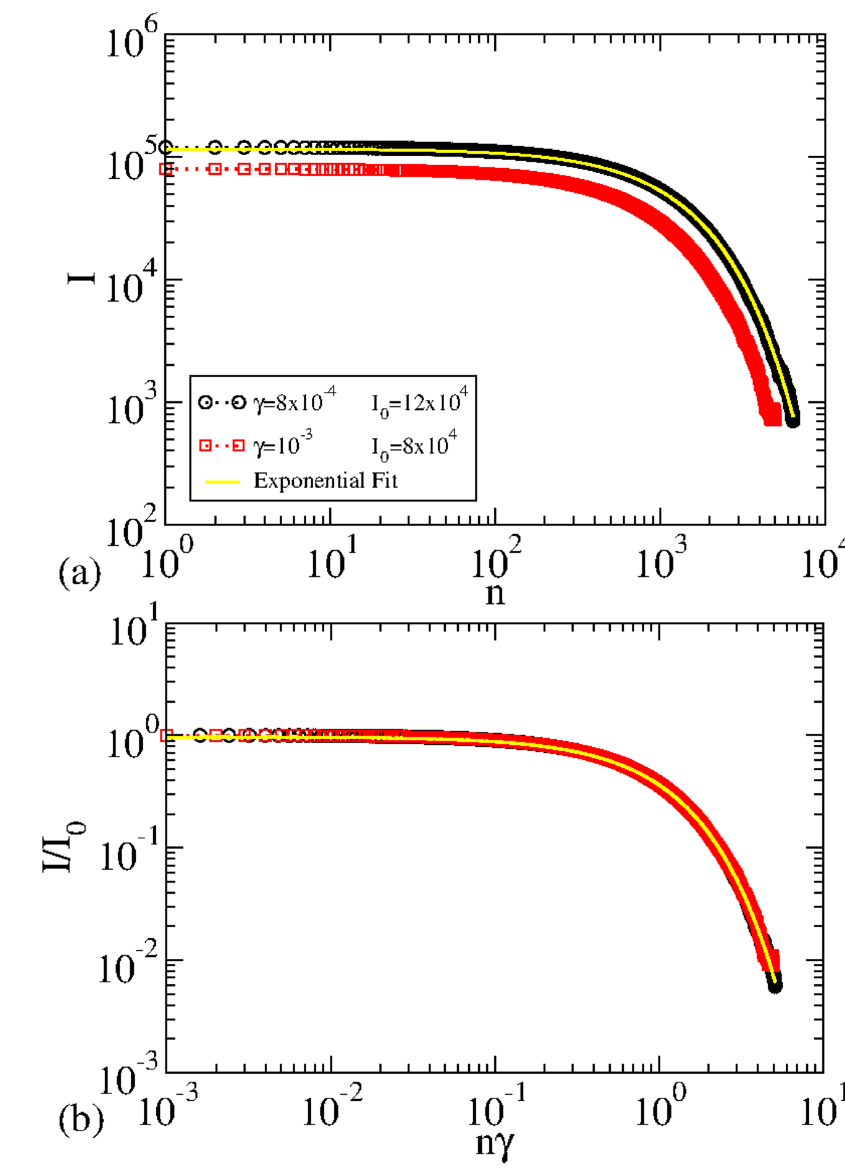

Figure 2: *Decay for the chaotic attractor using $k = 10$: (a) behavior of $I \times n$ for different initial conditions and values of γ (as labeled in the figure) and (b) overlap of all curves in a single universal curve.*

Figure 3 shows the behavior of the Lyapunov exponents of mapping (1). The control parameters used were $k = 10$ and $\gamma = 10^{-2}$ and a set of six different initial conditions, as labeled in the figure. Each initial conditions was iterated 10^8 times. Figure 3(a) shows the behavior of the positive Lyapunov exponents. An average over the set of different curves yields $\bar{\lambda}_1 = 2.00300(4)$. Figure 3(b) shows the behavior of the negative Lyapunov exponent where the average value gives $\bar{\lambda}_2 = -2.01304(4)$. Figure 3(c) shows the sum of the average values of the Lyapunov exponents $\bar{\lambda}_1 + \bar{\lambda}_2 \sim -10^{-2}$ which has the same magnitude than γ.

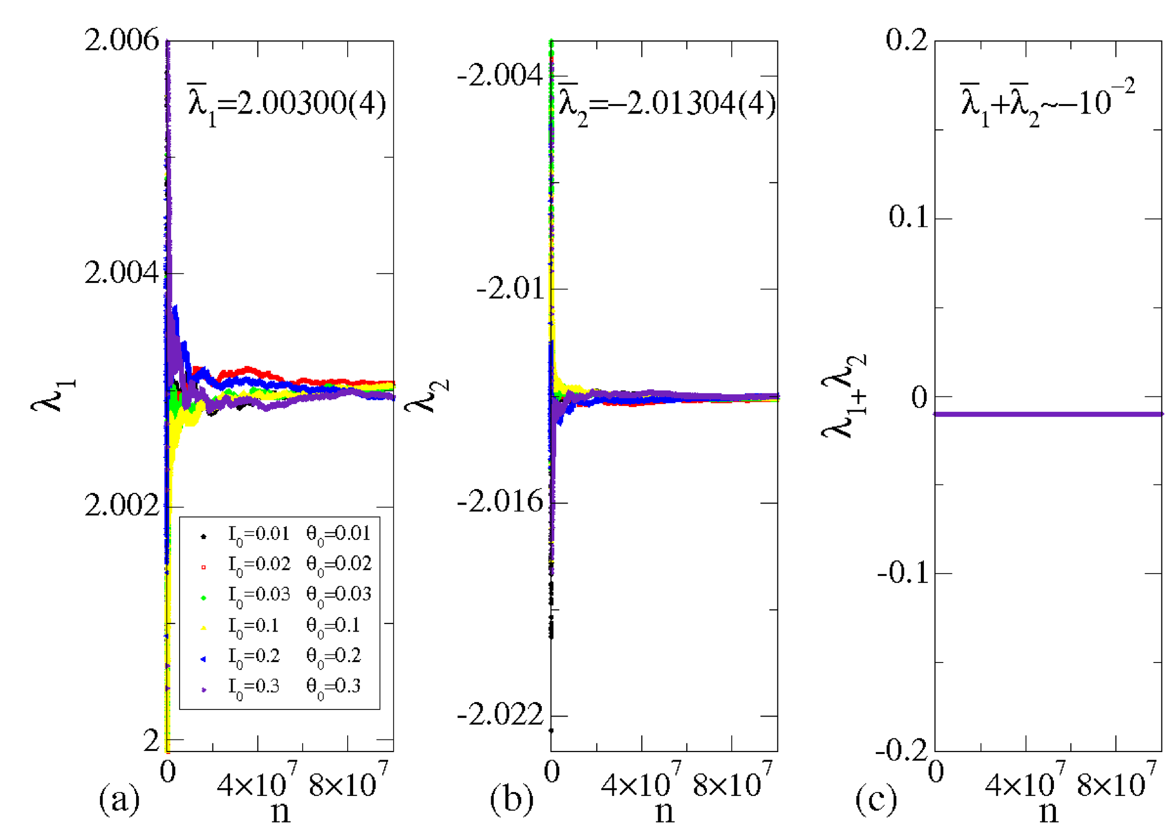

Figure 3: *Plot of the behavior of the (a) positive and (b) negative Lyapunov exponents for the control parameters $k = 10$ and $\gamma = 10^{-2}$. (c) Sum of the average values of the Lyapunov exponents $\bar{\lambda}_1$ and $\bar{\lambda}_2$.*

4. Conclusions

To summarize, we have studied in this work using extensive simulations, that when we apply control parameters of the intensity of nonlinearity and dissipation in a nonlinear two-dimensional mapping of a kicked rotor, we transform it in a discontinuous dissipative standard mapping and find a chaotic attractor when the variables of the action and angle are evolved over time. To confirm the intensity of chaos in the mapping we used the Lyapunov exponents.

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