Organization of the periodicity windows in a family of two-dimensional mappings

Juliano A. de Oliveira[*], L. T. Montero, D. R. da Costa, J. A. Mendez-Bermùdez, R. O. Medrano-T and E. D. Leonel

[1]Universidade Estadual Paulista (UNESP), Câmpus de São João da Boa Vista, SP, Brazil

[2]Universidade Estadual Paulista (UNESP), Instituto de Geociências e Ciências Exatas, Departamento de Física, Câmpus de Rio Claro, SP, Brazil

[3]Instituto de Física, Benemérita Universidad Autónoma de Puebla, México

[4]Universidade Federal de São Paulo (UNIFESP), Instituto De Ciências Ambientais, Químicas e Farmacêuticas, Departamento de Física, Câmpus de Diadema, Brazil

1 Abstract

- Hamilton Systems - a mathematical formalism developed by Hamilton to describe the evolution of equations of the physical system.
- Discrete mappings - the evolution equations allows us to calculate an state I_{n+1} at a time $n+1$ from the state I_n at the previous time n.
- Dissipation is introduced in the system.
- Lyapunov exponents are used to characterize the chaotic attractors and show the organization of the periodicity windows.

2 Motivation

We discuss some dynamical properties for a set of two dimensional Hamiltonian mappings. We assume that there is a two-dimensional integrable system that is slightly perturbed. The Hamiltonian function that describes the system is

$$H(I_1, I_2, \theta_1, \theta_2) = H_0(I_1, I_2) + \varepsilon H_1(I_1, I_2, \theta_1, \theta_2)\ , \tag{1}$$

where the variables I_i and θ_i with $i = 1, 2$ correspond respectively to the action and angle and ε controls a transition from integrability to non integrability. A two dimensional mapping which qualitatively describes the behavior of (1) is

$$T : \begin{cases} I_{n+1} = I_n + \varepsilon H(\theta_n, I_{n+1}) \\ \theta_{n+1} = [\theta_n + F(I_{n+1}) + \varepsilon P(\theta_n, I_{n+1})] \mod(2\pi) \end{cases} \tag{2}$$

where H, F and P are assumed to be nonlinear functions of their variables.

For many mappings in the literature, the function $P(\theta_n, I_{n+1}) = 0$. Hence, if we keep H as $H(\theta_n) = \sin(\theta_n)$, and vary F, to illustrate applicability of the formalism, we nominate the following mappings:

- Considering $F(I_{n+1}) = I_{n+1}$, the Taylor-Chirikov's map is obtained;
- $F(I_{n+1}) = 2/I_{n+1}$, the Fermi-Ulam accelerator model is recovered;
- $F(I_{n+1}) = \zeta I_{n+1}$, with ζ constant, the bouncer model is obtained;
- $F(I_{n+1}) = 1/I_{n+1}^{3/2}$, the kepler map is found.

3 The Model

We consider the introduction of dissipation in a family of two-dimensional mappings. The mappings are defined as

$$T : \begin{cases} I_{n+1} = |\delta I_n - (1+\delta)\epsilon \sin(2\pi\theta_n)| \\ \theta_{n+1} = \left[\theta_n + I_{n+1}^{\gamma}\right] \mod 1 \end{cases}\ , \tag{3}$$

where the parameter ϵ controls the non-linearity, δ is the parameter controlling the amount of dissipation and γ is a free parameter. If $\delta = 1$ the conservative case is recovered.

In the dissipative case the determinant of the Jacobian matrix of the mapping (3) is $\mathrm{Det\ J} = \delta\ \mathrm{sign}\left[\delta I_n - (1+\delta)\epsilon \sin(2\pi\theta_n)\right]$, where the function $\mathrm{sign}(u) = 1$ if $u > 0$ and $\mathrm{sign}(u) = -1$ if $u < 0$.

4 Parameter Space

To explore the influence of the dissipation in the parameter space δ vs. ϵ, we use two techniques: (i) calculation of the maximum Lyapunov exponent, and (ii) computation of periods. While in the latter, the period is directly computed by counting the number of points that compose the attractor. The Lyapunov exponent were computed as

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \ln \left|\Lambda_j^{(n)}\right|, \quad j = 1, 2, \ldots \tag{4}$$

where $\Lambda_j^{(n)}$ are the eigenvalues of the matrix $M = \Pi_{i=1}^{n} J_i(\theta, I)$ and J_i is the Jacobian matrix of the mapping evaluated along the orbit (θ_i, I_i). If at least one λ_j is positive, then the system is withing a chaotic regime. On the other hand, when trajectories are describing periodic oscillations, all λ_j are negative. $\lambda = 0$ indicates the attractor is undergoing a bifurcation.

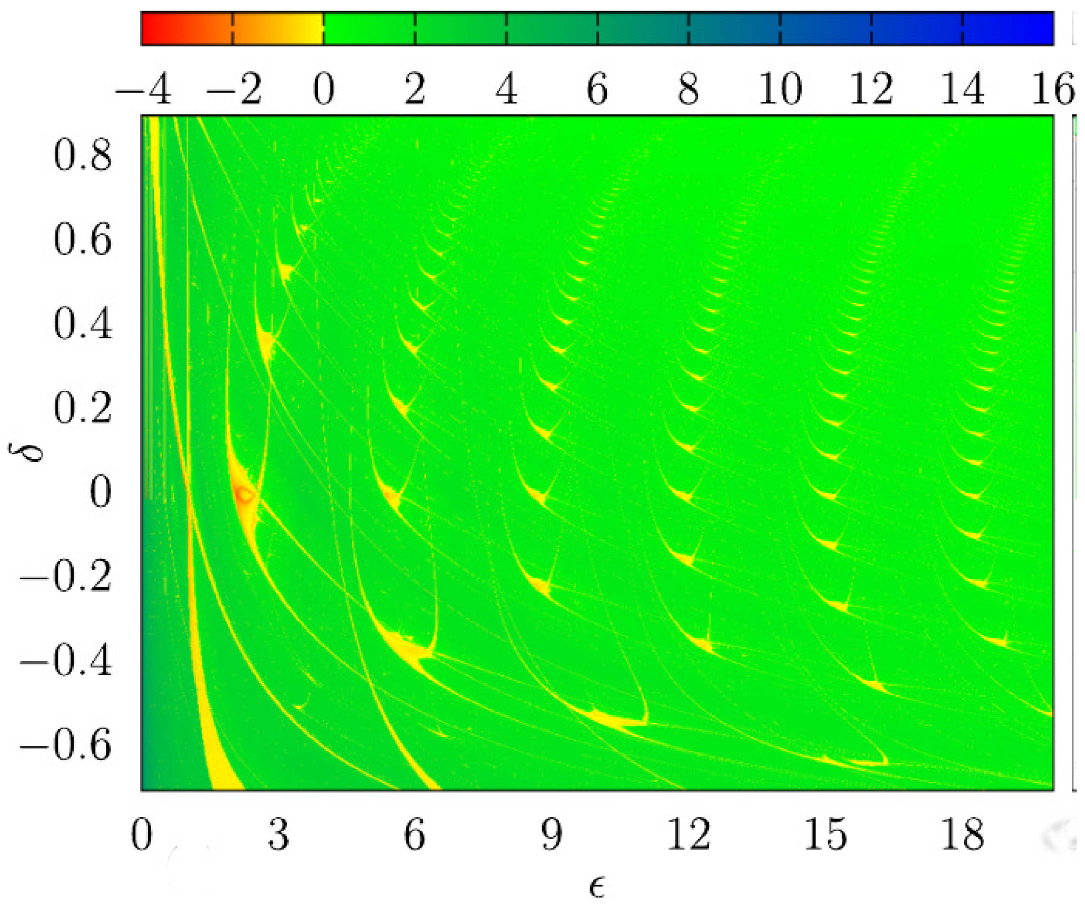

Fig. 1: *Parameter space* δ vs. ϵ *for* $\gamma = -1$*. The color palette indicates the maximum Lyapunov exponents. Source [7].*

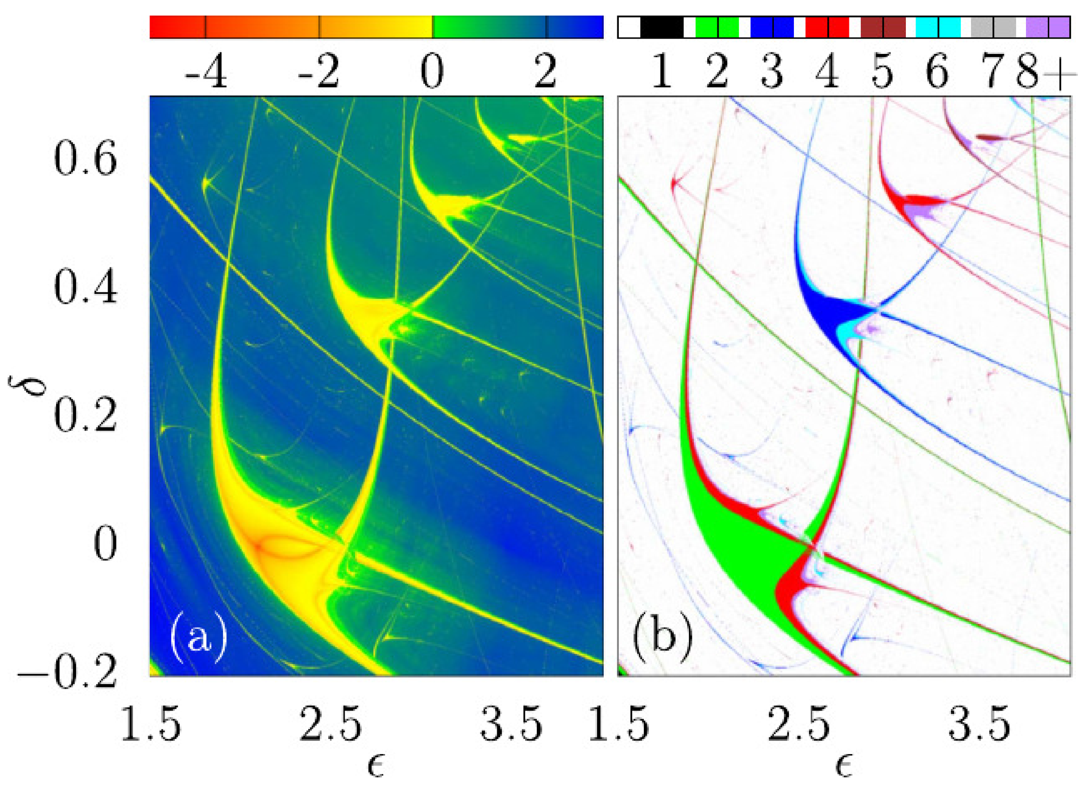

Fig. 3: *(a) Sequence of slightly different complex sets colored according to the maximum Lyapunov exponent and (b) the corresponding period of the orbits. Source [7].*

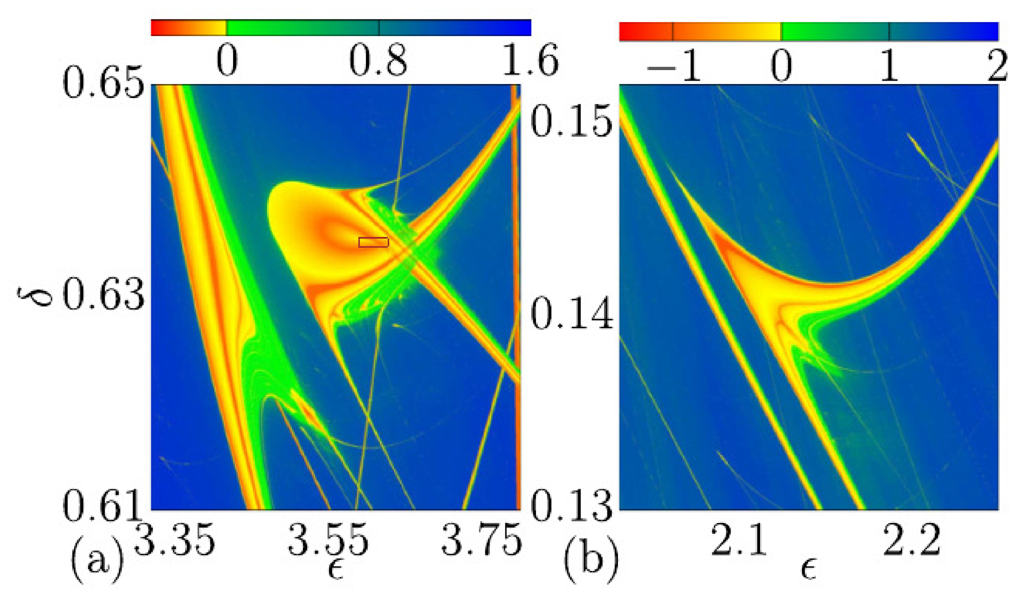

Fig. 4: *Structures resulting from a cubic homoclinic tangencies colored by the Lyapunov exponent: (a) spring-area and (b) saddle-area. Source [7].*

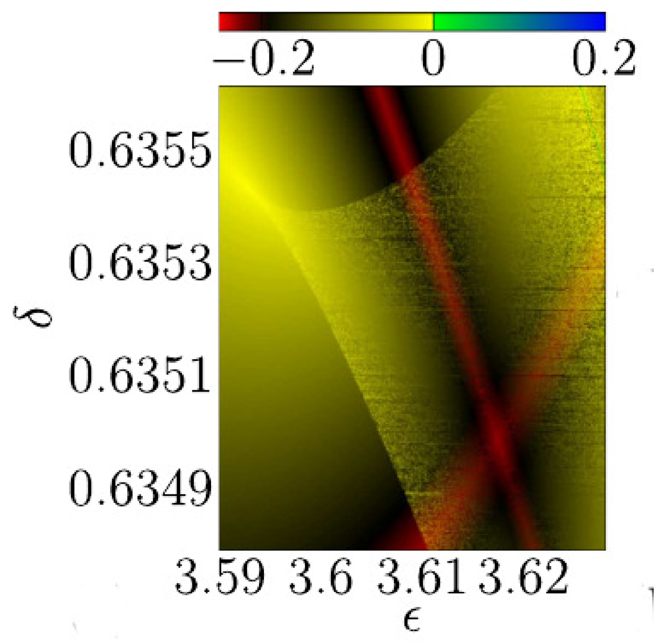

Fig. 5: *Magnification of the region delimited by a rectangle in Fig. 4(a). Source [7].*

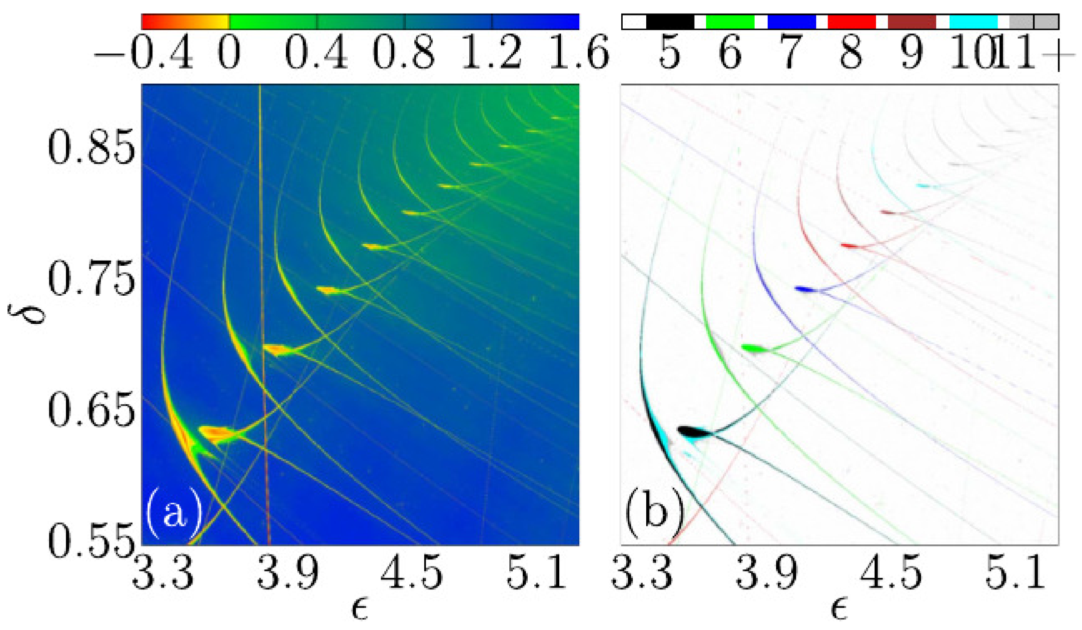

Fig. 6: *Parameter space* δ vs. ϵ *highlighting part of a street shown in Fig. 1 colored by the Lyapunov exponent while in (b) the color identifies the period of the structure. Source [7].*

5 Final Comments

We have considered a family of two-dimensional mappings parametarized by ϵ and δ control parameter. The choose of the control parameter $\delta = 1$ recover the conservative system. We used the Lyapunov exponents to characterize the chaos and by observing structures of periodicity in the high definition parameter space, we identified the shrimps structures.

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*juliano.antonio@unesp.br