Analytical Solution of The Diffusion Equation in Hamiltonian Mappings

Célia Mayumi Kuwana*, Edson Denis Leonel

Universidade Estadual Paulista "Júlio de Mesquita Filho" - *campus* Rio Claro

⋆celiamkuwana@hotmail.com

1st Perspectives on Oscillation Control

Introduction

In this work, we consider a discrete mapping described by *momentum*, I, and generalized coordenate, θ, and controled by two parameters: ε, tunning the intensity of nonlinearity, and γ, describing the form of divergence of θ when $I \to 0$.

The goal of this work is to describe the curves of average *momentum*, $I_{rms}(n)$, in terms of n, from a probability function, $P(I(n))$, to observe a given *momentum* I at an instant n. Therefore, we will solve the Diffusion equation analitically considering the cases: *(i)* the initial action is null, $I_0 = 0$, and *(ii)* the initial action is nonzero, $I_0 \neq 0$.

Results and Discussions

The mapping studied is given by:

$$T : \begin{cases} I_{n+1} = & I_n + \varepsilon \sin(\theta_n) \\ \theta_{n+1} = & \left(\theta_n + \frac{1}{|I_{n+1}|^{\gamma}}\right) \ mod(2\pi) \end{cases}, \quad (1)$$

where ε is a parameter that controls the transition from integrabiliy, when $\varepsilon = 0$, to nonintegrability, when $\varepsilon \neq 0$, and $\gamma > 0$ is a free parameter that controls the behavior of θ_{n+1} in the limit where $I_{n+1} \to 0$.

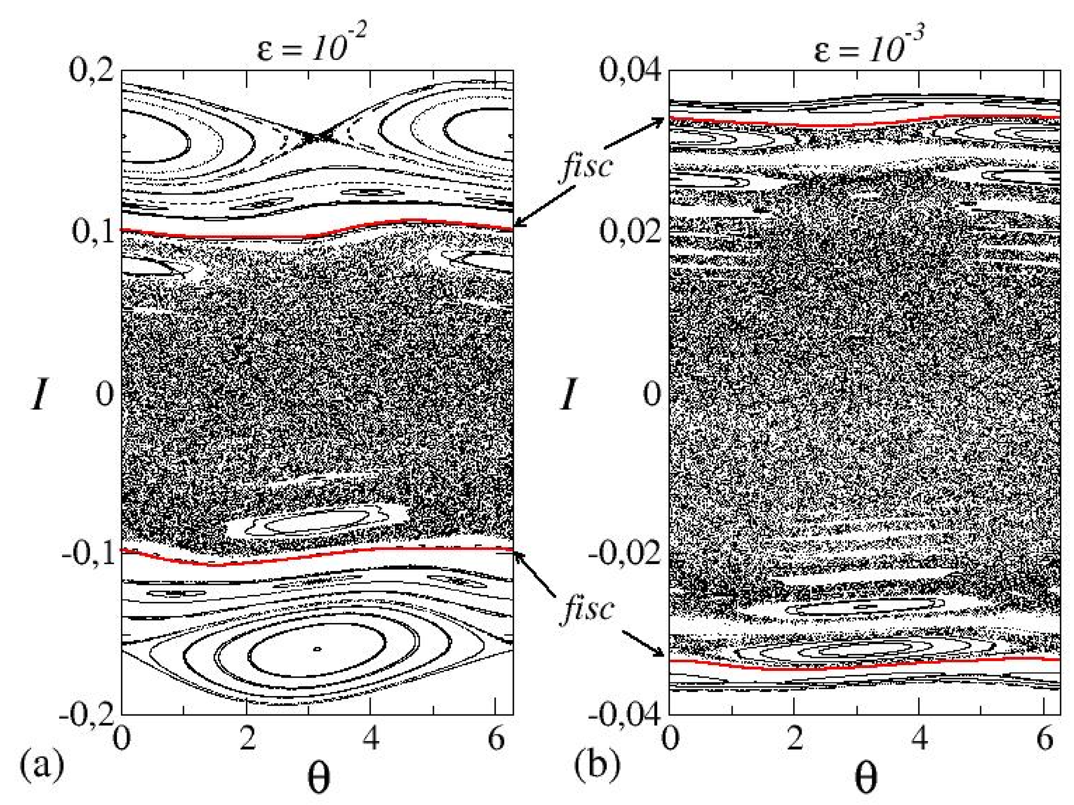

Figure 1: *Phase space of (1) with* $\gamma = 1$, *(a)* $\varepsilon = 10^{-2}$ *and (b)* $\varepsilon = 10^{-3}$. *The invariant spanning curves are represented by fisc and shown in red color.*

The location of the first invariant *spanning* curves is obtained connecting the mapping (1) with Chirikov-Taylor mapping and we obtain that $I_{fisc} = \left[\frac{\gamma}{K_{ef}}\right]^{\frac{1}{\gamma+1}} \varepsilon^{\frac{1}{\gamma+1}}$, where K_{ef} denotes the transition from local chaos to global chaos. The observable of interest is defined as $I_{rms}(n) = \sqrt{\overline{I^2}(n)}$, where:

$$\overline{I^2}(n) = \frac{1}{M}\sum_{i=1}^{M}\frac{1}{n}\sum_{j=1}^{n} I_{i,j}^2, \quad (2)$$

M corresponds to the ensemble average of $\theta \in [0, 2\pi]$ and n is the number of iterations of the mapping.

The dynamics in the chaotic sea can be compared to a normal diffusive process, so we solve the diffusion equation that, without external fields, is given by:

$$\frac{\partial P(I,n)}{\partial n} = D\frac{\partial^2 P(I,n)}{\partial I^2}, \quad (3)$$

where D is the diffusion coefficient obtained from $D = \frac{(\Delta \overline{I^2})}{2} = \overline{I^2}_{n+1} - \overline{I^2}_n = \frac{\varepsilon^2}{4}$ and $P(I,n)$ is the probability of observing a specific *momentum* I in an instant n.

The boundary conditions for the problem are given by $\left.\frac{\partial P}{\partial I}\right|_{I=\pm I_{fisc}} = 0$, that is, there is no flux of particles through the invariant *spanning* curves.

The procedure used to obtain a possible solution of diffusion equation is the separation of variables writing $P(I,n) = X(I)N(n)$, where $X(I)$ is a function that depends only on I and $N(n)$ is another function that depends only on n. After some algebra, we have:

$$P[I(n)] = \frac{1}{2I_{fisc}} + \frac{1}{I_{fisc}}\sum_{k=1}^{\infty}\cos\left[\frac{k\pi(I - I_0)}{I_{fisc}}\right] e^{-\frac{k^2\pi^2 Dn}{I_{fisc}^2}}. \quad (4)$$

where I_0 defines the initial *momentum* along the chaotic sea, n corresponds to the iteration number of the mapping and k comes from the boundary conditions.

Once the phase space is symmetric with respect to I, the observable studied is $\overline{I^2}(n)$ instead $\overline{I}(n)$. This observable expression can be obtained from $\overline{I^2} = \int_{-I_{fisc}}^{I_{fisc}} I^2 P[I(n)]dI$, which leads to:

$$\overline{I^2} = I_{fisc}^2\left[\frac{1}{3} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\left(\frac{k\pi I_0}{I_{fisc}}\right) e^{-\frac{k^2\pi^2 Dn}{I_{fisc}^2}}\right]. \quad (5)$$

So:

$$I_{rms}(n) = I_{fisc}\sqrt{\frac{1}{3} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\left(\frac{k\pi I_0}{I_{fisc}}\right) e^{-\frac{k^2\pi^2 D}{I_{fisc}^2}}\left[\frac{1}{n}\left(\frac{1 - e^{-\frac{k^2\pi^2 Dn}{I_{fisc}^2}}}{1 - e^{-\frac{k^2\pi^2 D}{I_{fisc}^2}}}\right)\right]}. \quad (6)$$

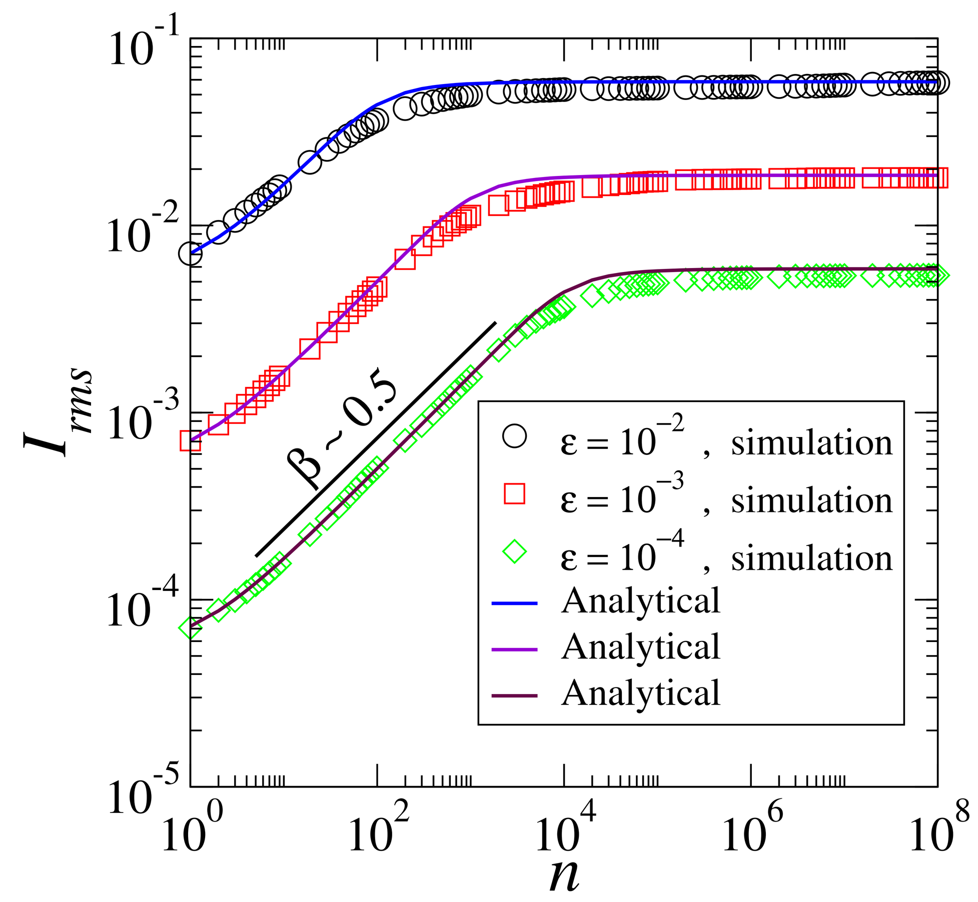

Figure 2: *Plot of* I_{rms} *vs.n with* $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$ *and* $\gamma = 1$ *and* $I_0 = 0$. *Symbols represent numerical simulations and continuous lines represent theoretical results.*

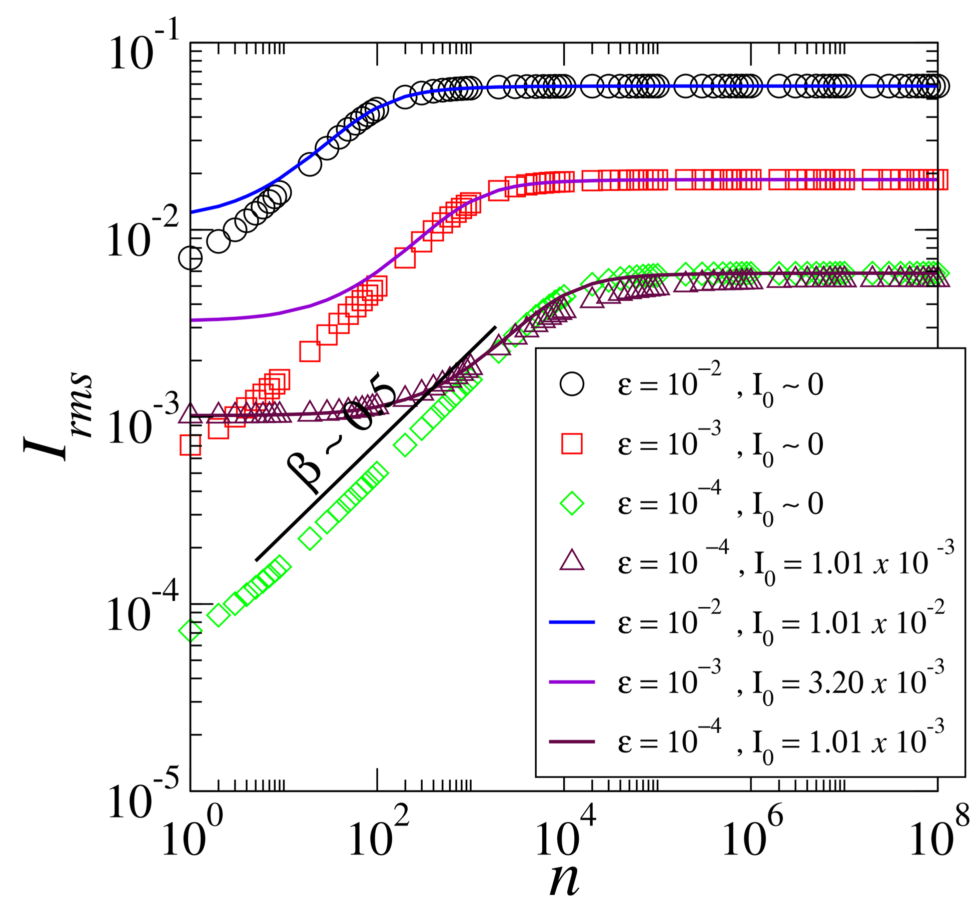

Figure 3: *Plot of* I_{rms} *vs.n with* $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$ *and* $\gamma = 1$ *and different values of* I_0. *Symbols represent numerical simulations and continuous lines represent theoretical results.*

Conclusions

From the diffusion equation, we obtained a solution analytically using the method of separation of variables, considering the cases that initial *momentum* is null and initial *momentum* different from zero. This is our original contribution for the problem.

References

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