Fractal structures in the parameter space of nontwist area-preserving maps

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1st Perspectives on Oscillation Control

Introduction

Fractal structures are very common in the phase space of nonlinear dynamical systems[1] and can be related to the final state uncertainty with respect to small perturbations on initial conditions[2]. In this work we investigate fractal structures in the parameter space of the standard nontwist map (SNTM)[3].

The violation of the twist property in SNTM leads to the existence of a shearless invariant curve, which acts as a barrier separating chaotic regions in the phase space[4]. This internal transport barrier creates two different behaviors, namely the escape of trajectories to plus or minus infinity[5]. The parameter space of the standard nontwist map presents an involved boundary between these behaviors[4].

In this work we present two different quantitative characterization of the boundary in parameter space. The first is the computation of the uncertainty dimension[2]. The second is the determination of the basin entropy and basin boundary entropy to quantify the degree of uncertainty due to the fractality of the boundary[6].

Results and Discussions

The standard nontwist map (SNTM) was defined in Ref. [3]

$$x_{n+1} = x_n + a\left(1 - y_{n+1}^2\right),$$
$$y_{n+1} = y_n - b\sin\left(2\pi x_n\right),$$

where $x \in [-1/2, 1/2)$ and $y \in \mathbb{R}$ are coordinates in the phase space $\mathbb{T} \times \mathbb{R}$, and $a \in (0, 1)$ and $b \in \mathbb{R}$ are parameters of the system.

In figure bellow, which focus on the boundary part [see Fig. 11 of Ref. [5], we show magnifications of the regions labeled as 1 to 5 in Fig. (a).

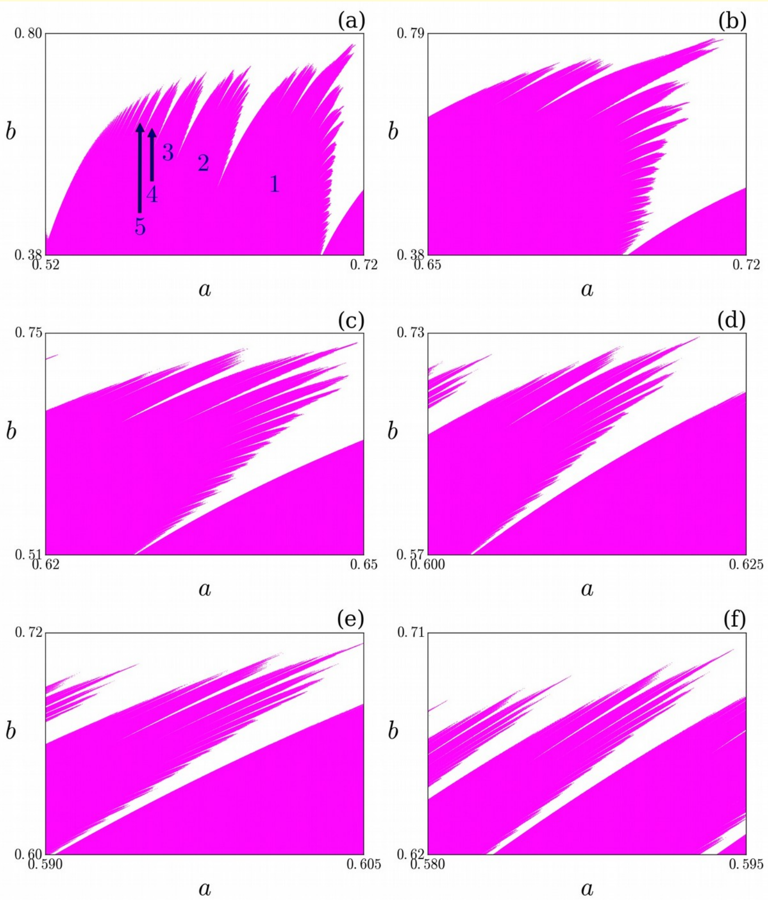

FIG. (a) Region, in the parameter space of the SNTM, for which the shearless curve exists (magenta points). White points represent parameter values for which the orbits escape to plus or minus infinity. (b)-(f): A sequence of magnifications of the regions specified by numbers (1)-(5) in (a).

A cursory inspection of Fig. suggests that the boundary between the parameter regions for which a shearless invariant curve exists (magenta) and does not exist is self-similar and has a fractal structure. Actually the boundary cannot be completely fractal, since there are smooth parts of it corresponding to bifurcation curves, as shown by Wurm and et al[5].

Conclusions

We used the method of uncertainty exponent to estimate the dimension of the boundary in the parameter space. We obtained the value of ~ 1.8 for the dimension of the boundary as whole. Moreover this value is consistently the same in different regions of the boundary, suggesting that it is a homogeneous fractal structure.

We also computed the basin entropy (S_b) and basin boundary entropy (S_{bb}), which quantify the uncertainty of the final state of any dynamical system. We obtained values of S_b ~ 0.05 and S_{bb} ~ 0.45, which are also characteristic of a complicated fractal basin structure.

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Acknowledgment: FAPESP FUNDAÇÃO DE AMPARO À PESQUISA DO ESTADO DE SÃO PAULO, CAPES, CNPq