Variational Principle

- **Hamilton's variational principle**

$$\delta \int_{t_1}^{t_2} dt\; L(q,\dot{q},t) = 0 \tag{1}$$

For a massless particle under a pure magnetic field,

$$L = \frac{1}{2}\cancelto{0}{m}\, v^2 - e\,\cancelto{0}{\Phi} + e\mathbf{A}\cdot\mathbf{v} \qquad \rightarrow \qquad L = e\mathbf{A}\cdot\mathbf{v} \tag{2}$$

$$\delta \int_{t_1}^{t_2} dt\; L(q,\dot{q},t) = 0 \quad \rightarrow \quad \delta \int_{t_1}^{t_2} \mathbf{A}\cdot\mathbf{v}\, dt = 0 \quad \rightarrow \quad \delta \int_{t_1}^{t_2} \mathbf{A}\cdot\frac{d\mathbf{r}}{dt}\, dt = 0 \tag{3}$$

$$\delta \int_{r_1}^{r_2} \mathbf{A}\cdot d\mathbf{r} = 0 \tag{4}$$

- **Hamiltonian description**

In curvilinear coordinates, the components of $\mathbf{B} = \nabla\times\mathbf{A}$ are,

$$B^1 = \frac{1}{\sqrt{g}}\left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\right), \qquad B^2 = \frac{1}{\sqrt{g}}\left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right), \qquad B^3 = \frac{1}{\sqrt{g}}\left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\right) \tag{5}$$

With the chosen gauge $A_2 = 0$, we have,

$$\begin{aligned} B^1 &= \frac{1}{\sqrt{g}}\frac{\partial A_3}{\partial x^2} &&\rightarrow \quad A_3 = \int \sqrt{g}B^1 dx^2 \\ B^3 &= \frac{1}{\sqrt{g}}\left(-\frac{\partial A_1}{\partial x^2}\right) &&\rightarrow \quad A_1 = -\int \sqrt{g}B^3 dx^2. \end{aligned} \tag{6}$$

From equations (18)-(21),

$$\begin{aligned} x^1 &= q &&\rightarrow \quad q = x^1 \\ x^3 &= t &&\rightarrow \quad t = x^3 \\ A_1 &= p &&\rightarrow \quad p = -\int \sqrt{g}\, B^3 dx^2 \\ A_3 &= -H &&\rightarrow \quad H = -\int \sqrt{g}\, B^1 dx^2 \end{aligned} \tag{7}$$

So, B^2 is defined as,

$$B^2 = \frac{1}{\sqrt{g}}\left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\right) \quad \rightarrow \quad B^2 = \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial t}p - \frac{\partial}{\partial q}(-H)\right) \tag{8}$$

$$\sqrt{q}B^2 = \frac{\partial p}{\partial t} + \frac{\partial H}{\partial q} \tag{9}$$