Pendulum approximation

Resonance

The resonance occurs when,

$$\frac{d\theta}{d\phi} = \frac{n}{m} \quad \longrightarrow \quad \frac{d\phi}{d\theta} = \frac{m}{n} \tag{1}$$

The definition of the safety factor, in the large aspect ration approximation, is $q = d\phi/d\theta$. Therefore, in the resonance position r^*,

$$q(r^*) = \left.\frac{d\phi}{d\theta}\right|_{r^*} = \frac{m}{n}. \tag{2}$$

With this, we can calculate the respective action variable of the resonance position r^*, from the non-canonical transformation $J = r^2/2$. The safety factor related to H_0 is given by Equation (47) of the paper:

$$q(r) = q_a \left(2 - \frac{r^2}{a^2}\right)^{-1}. \tag{3}$$

In the resonance r^*, we obtain,

$$q(r^*) = q_a \left(2 - \frac{r^{*2}}{a^2}\right)^{-1} = \frac{m}{n}, \tag{4}$$

and the values of r^* and, consequently, J^* are,

$$r^* = \left[a^2\left(2 - \frac{n\, q_a}{m}\right)\right]^{1/2} \quad \text{and} \quad J^* = a^2\left(1 - \frac{n\, q_a}{2\, m}\right). \tag{5}$$

Limiter Hamiltonian in the resonance

The Hamiltonian associated with the contribution of the Ergodic Magnetic Limiter (EML) is given by equation (53) of the paper, the function

$$H_1(J,\theta,\phi) = -\sigma A_m(J)\left\{\cos(m\theta) + \sum_{n=1}^{\infty}\left[\cos(m\theta - n\phi) + \cos(m\theta + n\phi)\right]\right\}, \tag{6}$$

where,

$$\sigma = \frac{\mu_0\, I_L\, \ell}{2\,\pi^2\, B_0} \quad \text{and} \quad A_m(J) = \frac{(2J)^{m/2}}{a^m}. \tag{7}$$

Next to the exact resonance position J^*, the term $\cos(m\theta - n\phi)$ slowly oscillates and it is the only term that influences the system near the resonance. The other terms vanish if an average is performed over ϕ. Here, we demonstrate this result.

The average of the terms $\cos(m\theta \pm n\phi)$ over ϕ is,

$$\overline{\cos(m\theta \pm n\phi)} = \frac{1}{2\pi}\int_0^{2\pi} \cos(m\theta \pm n\phi) d\phi \tag{8}$$

In the resonance, $m\theta - n\phi$ is constant. So, considering $m\theta - n\phi = \alpha$ a constant, the average is,

$$\overline{\cos(m\theta - n\phi)}|_{m\theta - b\phi = \alpha} = \frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha) d\phi = \frac{\cos\alpha}{2\pi} 2\pi = \cos\alpha. \tag{9}$$

For the other terms $\cos(m\theta \pm n\phi)$, the average over ϕ is,

$$\overline{\cos(m\theta \pm n\phi)} = \frac{1}{2\pi}\int_0^{2\pi} \cos(m\theta \pm n\phi) d\phi = \pm\frac{1}{2\pi n}[\sin(m\theta \pm 2\pi n) - \sin(m\theta)] = 0. \tag{10}$$

As mentioned, the term $\cos(m\theta - n\phi)$ is the only term that does not vanish in the average performed over ϕ.

The contribution of EML near the resonance associated with n and m is,

$$H_1(J,\theta,\phi) = -\sigma A_m(J)\ \cos(m\theta - n\phi). \tag{11}$$

We omit the term $\cos(m\theta)$ once we are not interested in the resonance associated with $n = 0$.

The complete Hamiltonian function, for the system near the resonance, is

$$H_{res} = H_0(J) - \sigma A_m(J)\cos(m\theta - n\phi) \tag{12}$$

In the vicinity of the resonance $J = J^*$, we have a small $\Delta\, J = J - J^*$. Expanding H_{res} around the resonance, we have

$$\begin{aligned} H_{res} &= H_{res}(J^*) + \left.\frac{\partial H_{res}}{\partial J}\right|_{J^*} \Delta J + \frac{1}{2}\left.\frac{\partial^2 H_{res}}{\partial J^2}\right|_{J^*} (\Delta J)^2\ +\ \ldots \\ H_{res} &= H_0(J^*) - \sigma A_m(J^*)\cos(m\theta - n\phi) + \left(\frac{\partial H_0}{\partial J}\right)_{J^*} \Delta J - \sigma\left(\frac{\partial A_m}{\partial J}\right)_{J^*} \cos(m\theta - n\phi)\Delta J + \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 H_0}{\partial J^2}\right)_{J^*} (\Delta J)^2 + \frac{1}{2}\sigma\left(\frac{\partial^2 A_m}{\partial J^2}\right)_{J^*} \cos(m\theta - n\phi)(\Delta J)^2\ +\ \ldots \end{aligned} \tag{13}$$

Since σ and ΔJ assume small values, we assume the terms $\sigma\Delta J$ and $\sigma(\delta J)^2$ are small and we disregard them. With this condition, we obtain,

$$H_{res} = H_0(J^*) - \sigma A_m(J^*)\cos(m\theta - n\phi) + \left(\frac{\partial H_0(J)}{\partial J}\right)_{J^*} \Delta J + \frac{1}{2}\left(\frac{\partial^2 H_0(J)}{\partial J^2}\right)_{J^*} (\Delta J)^2. \tag{14}$$

Defining, $\Delta H(\Delta J,\theta,\phi) = H_{res} - H_0(J^*)$

$$\Delta H(\Delta J,\theta,\phi) = \left(\frac{\partial H_0(J)}{\partial J}\right)_{J^*} \Delta J + \frac{1}{2}\left(\frac{\partial^2 H_0(J)}{\partial J^2}\right)_{J^*} (\Delta J)^2 - \sigma A_m(J^*)\cos(m\theta - n\phi) \tag{15}$$

The Equation (48) of the paper inform us,

$$H_0(J) = \frac{2J}{q_a}\left(1 - \frac{J}{2a^2}\right). \tag{16}$$

Therefor, the respective derivatives at J^* are,

$$\left(\frac{\partial H_0}{\partial J}\right)_{J^*} = \frac{2}{q_a}\left(1 - \frac{J^*}{a^2}\right) = \frac{1}{q(J^*)} = \frac{1}{m/n} \quad \rightarrow \quad \boxed{\left(\frac{\partial H_0}{\partial J}\right)_{J^*} = \frac{n}{m}.} \tag{17}$$

$$\boxed{\left(\frac{\partial^2 H_0}{\partial J^2}\right)_{J^*} = -\frac{2}{q_a\, a^2}.} \tag{18}$$

Thus,

$$\Delta H(\Delta J, \theta, \phi) = \frac{n}{m}\Delta J - \frac{(\Delta J)^2}{q_a\, a^2} - \sigma A_m(J^*)\cos(m\theta - n\phi). \tag{19}$$

• **Canonical transformation**

Performing the canonical transformation $(\Delta J, \theta, \phi) \rightarrow (I, \psi)$ using the generating function $F_2(I, \theta, \phi) = (m\theta - n\phi)I$, we obtain,

$$\begin{aligned} \psi &= \frac{\partial F_2}{\partial I} = (m\theta - n\phi), \\ \Delta J &= \frac{\partial F_2}{\partial \theta} = m\, I, \\ \mathcal{H}(I, \psi) &= \Delta H(\Delta J, \theta, \phi) + \frac{\partial F_2}{\partial \phi} = \Delta H(\Delta J, \theta, \phi) - n\, I, \end{aligned} \tag{20}$$

which results in,

$$\mathcal{H}(I, \psi) = -\frac{m^2}{q_a\, a^2}\, I^2 - \sigma A_m(J^*)\cos\psi. \tag{21}$$

Defining,

$$G = \frac{-2\, m^2}{q_a\, a^2} \qquad \text{and} \qquad F = \sigma A_m(J^*), \tag{22}$$

the Hamiltonian function becomes,

$$\mathcal{H}(I, \psi) = \frac{1}{2} G I^2 - F\cos\psi, \tag{23}$$

the pendulum Hamiltonian.

■ **Half-width of a island**

The half-width of a island, in the pendulum approximation is $I_{max} = 2|F/G|^{1/2}$. For G and F defined by (22), we obtain

$$I_{max} = 2\left(\frac{\sigma A_m(J^*) q_a\, a^2}{2m^2}\right)^{1/2}. \tag{24}$$

With

$$\sigma = \frac{\mu_0\, I_L\, \ell}{2\pi^2\, B_0} = \varepsilon\xi\left(\frac{a^2}{q_a \pi}\right), \qquad A_m(J^*) = \frac{(2J^*)^{m/2}}{a^m}, \qquad J^* = a^2\left(1 - \frac{n\, q_a}{2\, m}\right), \tag{25}$$

we obtain,

$$I_{max} = \frac{2\, a^2}{m}\sqrt{\frac{\varepsilon\xi}{2\pi}}\left[2\left(1 - \frac{n\, q_a}{2\, m}\right)\right]^{m/4}. \tag{26}$$