Hamiltonian in cylindrical coordinates

From the equations (34) - (37),

$$q = \theta, \qquad p = -\int r\, B_z\, dr, \qquad t = z = R_0\, \phi, \qquad H = -\int B_\theta\, dr. \tag{1}$$

In the large aspect ratio approximation, we can consider for the equilibrium magnetic field,

$$B_r \;=\; 0, \qquad B_\theta \;=\; B_\theta(r), \qquad B_z \;=\; B_0 \;=\; const. \tag{2}$$

- Rotational transform ι and safety factor q:

$$\frac{2\pi}{\iota(r)} = q(r) = \frac{d\phi}{d\theta} = \frac{1}{R_0}\frac{dz}{d\theta}. \tag{3}$$

From the equation for the magnetic field lines,

$$\mathbf{B} \times \mathbf{r} = \mathbf{0} \tag{4}$$

$$\frac{dx^1}{B^1} = \frac{dx^2}{B^2} = \frac{dx^3}{B^3} \tag{5}$$

For cylindrical coordinates,

$$x^1 = \theta, \quad x^2 = r, \quad x^3 = z = R_0\phi, \tag{6}$$

$$rB^1 = B_\theta, \quad B^2 = B_r, \quad B^3 = B_z \tag{7}$$

Thus,

$$\frac{dx^1}{B^1} = \frac{dx^3}{B^3} \quad\rightarrow\quad \frac{d\theta}{B_\theta/r} = \frac{dz}{B_z} \quad\rightarrow\quad \frac{rd\theta}{B_\theta} = \frac{d(R_0\phi)}{B_z} \quad\rightarrow\quad \boxed{\frac{rd\theta}{B_\theta} = \frac{R_0 d\phi}{B_z}} \tag{8}$$

For the large aspect ratio approximation (2), we obtain

$$\frac{r\,d\theta}{B_\theta} = \frac{d\phi}{B_z}R_0 \quad\rightarrow\quad \frac{r\,d\theta}{B_\theta(r)} = \frac{R_0\,d\phi}{B_0} \tag{9}$$

$$q(r) = \frac{d\phi}{d\theta} = \frac{B_0\, r}{R_0\, B_\theta(r)}. \tag{10}$$

- Canonical momentum in the large aspect ratio approximation

$$p = -\int r\, B_z\, dr \quad \rightarrow \quad p = -\int r\, B_0\, dr \quad \rightarrow \quad \boxed{p = -\frac{B_0\, r^2}{2}} \tag{11}$$

- Hamiltonian function: $H = -\int B_\theta\, dr$

From (10),

$$\begin{aligned} q(r) = \frac{B_0\, r}{R_0\, B_\theta(r)} \quad &\rightarrow \quad B_\theta(r) = \frac{B_0\, r}{R_0\ q(r)}, \\ H = -\int \frac{B_0\, r}{R_0\ q(r)} d\, r &= -\frac{B_0}{R_0} \int \frac{r}{q(r)} dr. \end{aligned} \tag{12}$$

Following the large aspect approximation, from (11)

$$\begin{aligned} p = -\frac{B_0\, r^2}{2} \quad \rightarrow \quad dp = -B_0\, r\, dr \quad &\rightarrow \quad r\, dr = -\frac{dp}{B_0} \\ H = -\frac{B_0}{R_0} \int \frac{(-dp)}{B_0\, q(r)} \quad &\rightarrow \quad \boxed{H = \frac{1}{R_0} \int \frac{dp}{q(p)}} \end{aligned} \tag{13}$$

Considering the non-canonical transformation,

$$(p, q, t) \rightarrow \left(J = -\frac{p}{B_0}, \theta, \phi = \frac{t}{R_0} \right). \tag{14}$$

Thus,

$$J = -\frac{p}{B_0} \quad \rightarrow \quad dp = -B_0\, dJ, \tag{15}$$

$$H = \frac{1}{R_0} \int \frac{dp}{q(p)} \quad \rightarrow \quad H = -\frac{B_0}{R_0} \int \frac{dJ}{q(J)}. \tag{16}$$

Be a rescaled Hamiltonian function $H_0 = -R_0\, H / B_0$,

$$\boxed{H_0 = \int \frac{d\, J}{q(J)};} \tag{17}$$