Exploitation of the variational principle

- Variational principle

$$\delta \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{A}\cdot\mathbf{dr} = \int_1^2 \delta\,(\mathbf{A}\cdot\mathbf{dr}) = 0 \tag{1}$$

Once $\mathbf{A} = \nabla G + \psi\nabla\theta - \alpha\nabla\zeta$, we have

$$\int_1^2 \delta\,\mathbf{A}\cdot\mathbf{dr} = \int_1^2 \delta\,(\nabla G + \psi\nabla\theta - \alpha\nabla\zeta)\cdot\mathbf{dr} = 0, \tag{2}$$

$$\int_1^2 \delta\,(\nabla G\cdot\mathbf{dr} + \psi\nabla\theta\cdot\mathbf{dr} - \alpha\nabla\zeta\cdot\mathbf{dr}) = 0, \tag{3}$$

$$\int_1^2 \delta\,(dG + \psi d\theta - \alpha d\zeta) = 0, \tag{4}$$

$$\int_1^2 \delta\,\left(\frac{dG}{d\zeta}d\zeta + \psi\frac{d\theta}{d\zeta}d\zeta - \alpha\,d\zeta\right) = 0, \tag{5}$$

$$\tag{6}$$

Computing $\dfrac{d}{d\zeta}(\psi\theta)$,

$$\frac{d}{d\zeta}(\psi\,\theta) = \theta\,\frac{d\psi}{d\zeta} + \psi\,\frac{d\theta}{d\zeta} \quad\rightarrow\quad \psi\,\frac{d\theta}{d\zeta} = \frac{d}{d\zeta}(\psi\,\theta) - \theta\frac{d\psi}{d\zeta} \tag{7}$$

$$\psi\,\frac{d\theta}{d\zeta}\,d\zeta = \left[\frac{d}{d\zeta}(\psi\,\theta) - \theta\,\frac{d\psi}{d\zeta}\right]d\zeta\,. \tag{8}$$

$$\int_1^2 \delta\,\left(\frac{dG}{d\zeta}d\zeta + \psi\frac{d\theta}{d\zeta}d\zeta - \alpha\,d\zeta\right) = \int_1^2 \delta\,\left(\frac{dG}{d\zeta}d\zeta + \left[\frac{d}{d\zeta}(\psi\,\theta) - \theta\,\frac{d\psi}{d\zeta}\right]d\zeta - \alpha\,d\zeta\right) = 0, \tag{9}$$

$$\int_1^2 \delta\,\left(\frac{dG}{d\zeta} + \frac{d}{d\zeta}(\psi\,\theta) - \theta\,\frac{d\psi}{d\zeta} - \alpha\right)d\zeta = 0, \tag{10}$$

$$\int_1^2 \left[\delta\,\left(\frac{dG}{d\zeta}\right) + \delta\,\left(\frac{d}{d\zeta}(\psi\,\theta)\right) - \delta\,\left(\theta\,\frac{d\psi}{d\zeta}\right) - \delta\,\alpha\right]d\zeta = 0, \tag{11}$$

The variation of α for variations in ϕ and θ is $\delta\alpha = \dfrac{\partial\alpha}{\partial\psi}\,\delta\psi + \dfrac{\partial\alpha}{\partial\theta}\delta\theta$. The integral results in,

$$\int_1^2 \left[\delta\,\left(\frac{dG}{d\zeta}\right) + \delta\,\left(\frac{d}{d\zeta}(\psi\,\theta)\right) - \delta\,\left(\theta\,\frac{d\psi}{d\zeta}\right) - \frac{\partial\alpha}{\partial\psi}\,\delta\psi - \frac{\partial\alpha}{\partial\theta}\delta\theta\right]d\zeta = 0, \tag{12}$$

$$\int_1^2 \left[\frac{d}{d\zeta}\delta G + \delta\left(\theta\frac{d\psi}{d\zeta} + \psi\frac{d\theta}{d\zeta}\right) - \left(\delta\theta\,\frac{d\psi}{d\zeta} + \theta\delta\left(\frac{d\psi}{d\zeta}\right)\right) - \frac{\partial\alpha}{\partial\psi}\delta\psi - \frac{\partial\alpha}{\partial\theta}\delta\theta\right]d\zeta = 0, \tag{13}$$

$$\int_1^2 \left[\frac{d}{d\zeta}(\delta G) + (\delta\theta)\frac{d\psi}{d\zeta} + \theta\delta\left(\frac{d\psi}{d\zeta}\right) + (\delta\psi)\frac{d\theta}{d\zeta} + \psi\delta\left(\frac{d\theta}{d\zeta}\right) - (\delta\theta)\frac{d\psi}{d\zeta} - \theta\delta\left(\frac{d\psi}{d\zeta}\right) - \frac{\partial\alpha}{\partial\psi}\delta\psi - \frac{\partial\alpha}{\partial\theta}\delta\theta\right]d\zeta = 0, \tag{14}$$

The highlight terms can be rewritten as,

$$(\delta\theta)\frac{d\psi}{d\zeta} + \psi\delta\left(\frac{d\theta}{d\zeta}\right) = (\delta\theta)\frac{d\psi}{d\zeta} + \psi\left(\frac{d(\delta\theta)}{d\zeta}\right) = \frac{d}{d\zeta}(\psi\,\delta\theta). \tag{15}$$

So, the integer is reduced to,

$$\int_1^2 \left[\frac{d}{d\zeta}(\delta G) + \frac{d}{d\zeta}(\psi\, \delta\theta) + (\delta\psi)\frac{d\theta}{d\zeta} - (\delta\theta)\frac{d\psi}{d\zeta} - \frac{\partial\alpha}{\partial\psi}\delta\psi - \frac{\partial\alpha}{\partial\theta}\delta\theta\right] d\zeta = 0 \tag{16}$$

$$\int_1^2 \left[\frac{d}{d\zeta}(\delta G + \psi\delta\theta) + \left(\frac{d\theta}{d\zeta} - \frac{\partial\alpha}{\partial\psi}\right)\delta\psi - \left(\frac{d\psi}{d\zeta} + \frac{\partial\alpha}{\partial\theta}\right)\delta\theta\right] d\zeta = 0 \tag{17}$$

The first term vanishes,

$$\int_1^2 \frac{d}{d\zeta}(\delta G + \psi\delta\theta) d\zeta = [\delta G + \psi\delta\theta]_1^2 = (\delta G + \psi\delta\theta)_2 - (\delta G + \psi\delta\theta)_1 = 0. \tag{18}$$

since 1 and 2 are fixed points, and,

$$\int_1^2 \left[\left(\frac{d\theta}{d\zeta} - \frac{\partial\alpha}{\partial\psi}\right)\delta\psi - \left(\frac{d\psi}{d\zeta} + \frac{\partial\alpha}{\partial\theta}\right)\delta\theta\right] d\zeta = 0. \tag{19}$$

For arbitrary variation in ψ and θ, the integral vanishes if the coefficients vanish identically, namely,

$$\frac{d\theta}{d\zeta} - \frac{\partial\alpha}{\partial\psi} = 0, \qquad \frac{d\psi}{d\zeta} + \frac{\partial\alpha}{\partial\theta} = 0, \tag{20}$$

$$\frac{d\theta}{d\zeta} = \frac{\partial\alpha}{\partial\psi}, \qquad \frac{d\psi}{d\zeta} = -\frac{\partial\alpha}{\partial\theta}. \tag{21}$$