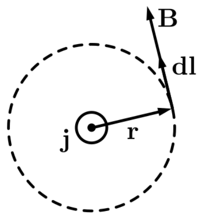
Example 1

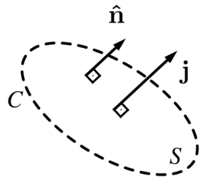
- Consider a plasma column of radius a, for which the electric current density $\mathbf{j}$ has radial symmetry with respect to the axis, carrying a total plasma current I_P. The component j_z is defined as,

$$j_z(r) = j_0\left(1-\frac{r^2}{a^2}\right) \tag{1}$$

where $j_0 = \dfrac{2\,I_P}{\pi\,a^2}$.

In order to obtain the Hamiltonian function associated, we firstly applied the Ampère's circuit law:





$$\oint_C \mathbf{B}\cdot\mathbf{dl} = \mu_0 \iint_S (\mathbf{j}\cdot\hat{\mathbf{n}})\,dS \tag{2}$$

$$\mathbf{B}\cdot\mathbf{dl} = B_\theta dl, \qquad \mathbf{j}\cdot\hat{\mathbf{n}} = j_z \tag{3}$$

$$\begin{aligned} \oint_C \mathbf{B}\cdot\mathbf{dl} = \mu_0 \iint_S (\mathbf{j}\cdot\hat{\mathbf{n}})\,dS \quad &\longrightarrow \quad \oint_C B_\theta(r)\,dl = \mu_0 \int_0^{2\pi}\int_0^r j_z\,r\,dr\,d\theta \\ B_\theta(r)\oint_C dl = \mu_0\,2\pi\int_0^r j_0\left(1-\frac{r^2}{a^2}\right)r\,dr \quad &\longrightarrow \quad \boxed{B_\theta(r) = \frac{\mu_0 j_0 r}{2}\left(1-\frac{r^2}{2a^2}\right) = \frac{\mu_0 I_P}{\pi a^2}r\left(1-\frac{r^2}{2a^2}\right)} \end{aligned} \tag{4}$$

In $r = a$,

$$B_\theta(r=a) = B_{\theta a} = \frac{\mu_0 I_P}{2\pi\,a}. \tag{5}$$

Therefore, the poloidal field radial profile is,

$$\boxed{B_\theta(r) = B_{\theta a}\frac{r}{a}\left(2-\frac{r^2}{a^2}\right).} \tag{6}$$

- **Safety factor profile**

From Equation (41),

$$q(r) = \frac{rB_0}{R_0 B_\theta(r)} \tag{7}$$

Using B_θ from (6),

$$q(r) = \frac{r\,B_0}{R_0}\left[B_{\theta a}\frac{r}{a}\left(2-\frac{r^2}{a^2}\right)\right]^{-1} = \frac{B_0\,a}{R_0 B_{\theta a}}\left(2-\frac{r^2}{a^2}\right)^{-1}. \tag{8}$$

From the definition of $B_{\theta a}$ of Equation (5), we obtain,

$$q(r) = \frac{2\pi\, a^2 B_0}{\mu_0\, R_0\, I_P} \left(2 - \frac{r^2}{a^2}\right)^{-1} \tag{9}$$

In $r = a$, $q(r = a) = q_a$ that is defined by,

$$q_a = \frac{2\pi\, a^2 B_0}{\mu_0\, R_0\, I_P}, \tag{10}$$

and the safety factor profile (9) can be written as,

$$q(r) = q_a \left(2 - \frac{r^2}{a^2}\right)^{-1}. \tag{11}$$

■ Hamiltonian function

The Hamiltonian function is given by Equation (44): $H_0 = \int \frac{dJ}{q(J)}$. From the non-canonical transformation, we have the relation $J = r^2/2$. Therefore, the profile (11) in function of the action J is given by,

$$q(J) = q_a \left(2 - \frac{2\,J}{a^2}\right)^{-1}, \tag{12}$$

and the Hamiltonian function is defined by,

$$H_0 = \int \frac{dJ}{q(J)} = \int \frac{dJ}{q_a \left(2 - \frac{2J}{a^2}\right)^{-1}} = \frac{1}{q_a}\left(2J - \frac{2J^2}{2a^2}\right) \tag{13}$$

$$H_0(J) = \frac{2J}{q_a}\left(1 - \frac{J}{2a^2}\right) \tag{14}$$