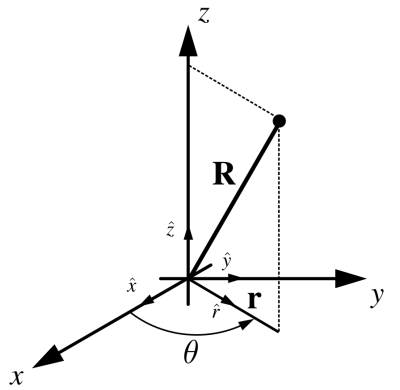
Cylindrical coordinates



Cartesian ⟷ Cylindrical

$x = r\ \cos\theta \qquad \mathbf{R} = \mathbf{r} + \mathbf{z}$

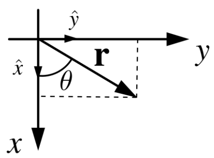
$y = r\ \sin\theta \qquad \mathbf{R} = r\,\hat{r} + z\hat{z}$

$z = z \qquad r = \sqrt{x^2 + y^2}$

If a vector $\mathbf{v}$ depends on u, the vector that points in the direction of increasing u is,

$$\mathbf{e}_u = \frac{\partial \mathbf{v}}{\partial u} \tag{1}$$

Identifying $x^1 = \theta$, $x^2 = r$ and $x^3 = z$, we compute the vector $\mathbf{e}_i$ in order to obtain the elements of the covariant metric tensor $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$.



$$\mathbf{r} = (\mathbf{r}\cdot\hat{\mathbf{x}})\ \hat{\mathbf{x}} + (\mathbf{r}\cdot\hat{\mathbf{y}})\ \hat{\mathbf{y}}$$

$$\mathbf{r} = r\ \cos\theta\ \hat{\mathbf{x}} + r\ \sin\theta\ \hat{\mathbf{y}}$$

Thus, $\mathbf{R} = r\ \cos\theta\ \hat{\mathbf{x}} + r\ \sin\theta\ \hat{\mathbf{y}} + z\,\hat{z}$, and the vectors $\mathbf{e}_i$ are,

$$\mathbf{e}_\theta = \frac{\partial \mathbf{R}}{\partial \theta} = \frac{\partial}{\partial \theta}(r\ \cos\theta\ \hat{\mathbf{x}} + r\ \sin\theta\ \hat{\mathbf{y}} + z\,\hat{z})$$

$$\mathbf{e}_\theta = -r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{y}} \tag{2}$$

$$\mathbf{e}_r = \frac{\partial \mathbf{R}}{\partial r} = \frac{\partial}{\partial r}(r\ \cos\theta\ \hat{\mathbf{x}} + r\ \sin\theta\ \hat{\mathbf{y}} + z\,\hat{z})$$

$$\mathbf{e}_r = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}} \tag{3}$$

$$\mathbf{e}_z = \frac{\partial \mathbf{R}}{\partial z} = \frac{\partial}{\partial z}(r\ \cos\theta\ \hat{\mathbf{x}} + r\ \sin\theta\ \hat{\mathbf{y}} + z\,\hat{z})$$

$$\mathbf{e}_z = \hat{\mathbf{z}} \tag{4}$$

The cylindrical coordinate system is orthogonal, therefore $g_{ij} = 0\ \forall\ i \neq j$. Accordingly,

$$g_{11} = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta = (-r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{y}}) \cdot (-r\sin\theta\,\hat{\mathbf{x}} + r\cos\theta\,\hat{\mathbf{y}}) = r^2\sin^2\theta + r^2\cos^2\theta = \quad \rightarrow \quad g_{11} = r^2$$

$$g_{22} = \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r = (\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}) \cdot (\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}) = \cos^2\theta + \sin^2\theta \quad \rightarrow \quad g_{22} = 1 \tag{5}$$

$$g_{33} = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} \quad \rightarrow \quad g_{33} = 1$$

The "physical" components, $B_{<i>} = \sqrt{g_{ii}}\, B^i$ become,

$$
\begin{aligned}
B_{<1>} = B_\theta = \sqrt{g_{11}} B^1 = \sqrt{r^2} B^1 \ &\rightarrow \ \boxed{B_{<1>} = r\, B^1} \\
B_{<2>} = B_r = \sqrt{g_{22}}\, B^2 = \sqrt{1} B^2 \ &\rightarrow \ \boxed{B_{<2>} = \ B^2} \\
B_{<3>} = B_z = \sqrt{g_{33}} B^3 = \sqrt{1} B^3 \ &\rightarrow \ \boxed{B_{<3>} = \ B^3}
\end{aligned}
\tag{6}
$$

Be the equations (29)-(32),

$$
q = x^1, \qquad p = -\int \sqrt{g}\, B^3\, dx^2, \qquad t = x^3, \qquad H = -\int \sqrt{g}\, B^1\, dx^2. \tag{7}
$$

where $g = g_{11}\, g_{22}\, g_{33}$. In cylindrical coordinates, $\sqrt{g} = \sqrt{r^2 \cdot 1 \cdot 1} = r$, $B^1 = B_\theta / r$, and,

$$
q = \theta, \qquad p = -\int r\, B_z\, dr, \qquad t = z = R_0\, \phi, \qquad H = -\int B_\theta\, dr. \tag{8}
$$